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NEW HERMITE–HADAMARD TYPE AND SOME RELATED INEQUALITIES VIA s–TYPE p–CONVEX FUNCTION

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Abstract. In this work, we define and introduce the idea of s–type p–convex function. We elaborate the new introduced idea by examples and some interesting algebraic properties. As a result, several new integral inequalities are established. These new results yield us some generalizations of the prior results.

1. Introduction

The theory of convexity has a rich and paramount history and has been the focus and fixed point of intense study for over a century in mathematics. This theory and their generalizations also play a magnificent role in the analysis of extremum problems. This theory had not only interesting and deep results in different branches of engineering and mathematical sciences, but due to widespread view and have a lot of applications, this theory provides numerical quadrature and amazing tools for researchers to tackle and to solve a wide class of related and unrelated problems.
The theory of convexity also played significant and meaningful role in the development of the theory of inequalities. Inequalities are one of the most important instrument in many branches of engineering and mathematics such as measure theory, probability theory, functional analysis, mathematical analysis, mechanics, physics and theory of differential and integral equations. Nowadays the theory of inequalities is still being intensively developed. Eventually the theory of inequalities may be regarded as an independent area of mathematics. İşcan et .al [11] first time introduced p–convex function. Motivated and inspired by the ongoing activities and research in the convex analysis field, we find out that there exists a special class of functions known as s–type convex function. Recently S. Rashid et .al [23] introduced n–polynomial s–type convex function.
The aim of this article is to define and introduce a new class of functions called s–type p–convex functions and also to study some of their algebraic properties. Several new inequalities via s–type p–convexity are establish. Examples with logic and applications via newly introduce definition are provided. The amazing techniques and remarkable ideas of this article may inspire and motivate for further research in this pivotal, captivating and valuable field. Before we start, we need the following necessary known definitions and literature.

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Definition 1. [18] *A function $\psi : I \to \mathbb{R}$ is said to be convex, if*

$$\psi\left(\kappa\varsigma_1 + (1-\kappa)\varsigma_2\right) \leq \kappa\psi\left(\varsigma_1\right) + (1-\kappa)\psi\left(\varsigma_2\right) \tag{1.1}$$

holds for all $\varsigma_1, \varsigma_2 \in I$ and $\kappa \in [0,1]$.

If the above inequality reverses, then ψ is said to be concave.

Many famously known results in inequalities theory can be obtained using the convexity property of the functions, see [5, 8, 14] and the references therein.

The Hermite–Hadamard inequality play an amazing and magnificent role in the literature, no one can refuse from its significance and importance. Since then researchers have shown keep interest in above inequality as a result various generalizations and improvements have been appeared in the literature. Many mathematicians put effort, hardworking and has collaborated different ideas and concepts in theory of inequalities and its applications. This type inequality has remained an area of great interest due to its widespread view and applications in the field of mathematical analysis. If a function $\psi : I \subset \mathbb{R} \to \mathbb{R}$ is convex in I for $\varsigma_1, \varsigma_2 \in I$ and $\varsigma_1 < \varsigma_2$, then

$$\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right) \leq \frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\kappa)d\kappa \leq \frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2}. \tag{1.2}$$

Interested readers can refer to [1]–[22].

Definition 2. [24] *A nonnegative function $\varphi : I \to \mathbb{R}$, is said to be h–convex, if*

$$\psi\left(\kappa\varsigma_1 + (1-\kappa)\varsigma_2\right) \leq h(\kappa)\psi\left(\varsigma_1\right) + h\left(1-\kappa\right)\psi\left(\varsigma_2\right), \tag{1.3}$$

holds for all $\varsigma_1, \varsigma_2 \in I$ and $\kappa \in (0,1)$.

ψ is h–concave, if above definition is reversed. Clearly that, if we substitute $h(\kappa) = \kappa$, then the h–convex functions collapses to the classical convex functions, see [2, 17].

Definition 3. [25] *If $I \subset (0,+\infty)$ be a real interval and $p \in \mathbb{R}\backslash 0$. A function $\psi : I \to \mathbb{R}$ is said to be a p–convex, if*

$$\psi\left(\left[\kappa\varsigma_1^p + (1-\kappa)\varsigma_2^p\right]^{\frac{1}{p}}\right) \leq \kappa\psi\left(\varsigma_1\right) + (1-\kappa)\psi\left(\varsigma_2\right), \tag{1.4}$$

$\forall\ \varsigma_1, \varsigma_2 \in I$ *and* $\kappa \in [0,1]$. *ψ is p–concave if the above inequality is reversed.*

According to the above definition, if we put $p = 1$, then p–convex functions collapses to ordinary convex functions defined in $I \subset (0,+\infty)$.

Definition 4. [23] *A function $\psi : I \to \mathbb{R}$, is said to be s–type convex function, if*

$$\psi\left(\kappa\varsigma_1 + (1-\kappa)\varsigma_2\right) \leq [1 - s(1-\kappa)]\psi(\varsigma_1) + [1 - s\kappa]\psi(\varsigma_2), \tag{1.5}$$

holds $\forall\ \varsigma_1, \varsigma_2 \in I$, $s \in [0,1]$ and $\kappa \in [0,1]$.

Motivated by the above results, literature and ongoing activities and research in this amazing and captivating field, we will give in Section 2, the idea and its algebraic properties of s–type p–convex function. In Section 3, we will derive the new version of Hermite–Hadamard inequality by using the newly introduced definition. As a result in Section 4, we will give related results. Finally, a brief conclusion will be provided as well.

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2. Some algebraic properties of s–type p–convex functions

We are going to add a new definition in this section namely s–type p–convex function and its some basic algebraic properties.

Definition 5. *A nonnegative function $\psi : I \to \mathbb{R}$, is said to be s–type p–convex, if*

$$\psi\left(\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{\frac{1}{p}}\right) \leq [1-(s(1-\kappa))]\psi\left(\varsigma_1\right) + [1-(s\kappa)]\psi\left(\varsigma_2\right), \tag{2.1}$$

holds for all $\varsigma_1, \varsigma_2 \in I$ and $\kappa \in [0,1]$.

Remark 1. *(i) Choosing $s = 1$, we get Definition 3.*
(ii) Choosing $p = 1$ in definition 5, we get Definition 4.
(iii) Choosing $p = -1$ in definition 5, we obtain the following new definition about harmonic s–type convex function:

$$\psi\left(\frac{\varsigma_1\varsigma_2}{\kappa\varsigma_2 + (1-\kappa)\,\varsigma_1}\right) \leq [1-(s(1-\kappa))]\psi\left(\varsigma_1\right) + [1-(s\kappa)]\psi\left(\varsigma_2\right). \tag{2.2}$$

(iv) Choosing $p = 1$ and $s = 1$ in definition 5, we get Definition 1.
(v) Choosing $p = -1$ and $s = 1$ in definition 5, we get Definition (2.1) in [13].

These are the best advantages of this newly introduce definition if we put the value of n, p and s , then we obtain new inequalities and also found some results which connect with previous results.

Lemma 1. *The following inequalities $[1-(s(1-\kappa))] \geq \kappa$ and $[1-(s\kappa)] \geq (1-\kappa)$ are hold. If for all $\kappa \in [0,1]$.*

Proof. The proof is evident. □

Proposition 1. *Let $I \subset (0, +\infty)$ be a p–convex set. Every p–convex function on a p–convex set is an s–type p–convex function.*

Proof. Using the definition of p–convex function and from the lemma 1 , since $\kappa \leq [1-(s(1-\kappa))]$ and $(1-\kappa) \leq [1-(s\kappa)]$ for all $\kappa \in [0,1]$, we have

$$\begin{aligned}\psi\left(\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{\frac{1}{p}}\right) &\leq \kappa\psi\left(\varsigma_1\right) + (1-\kappa)\psi\left(\varsigma_2\right)\\ &\leq [1-(s(1-\kappa))]\psi\left(\varsigma_1\right) + [1-(s\kappa)]\psi\left(\varsigma_2\right).\end{aligned}$$

□

Proposition 2. *Every s–type p–convex function is an h–convex function with $h(\kappa) = [1-(s(1-\kappa))]$.*

Proof.

$$\begin{aligned}\psi\left(\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{\frac{1}{p}}\right) &\leq [1-(s(1-\kappa))]\psi\left(\varsigma_1\right) + [1-(s\kappa)]\psi\left(\varsigma_2\right)\\ &\leq h(\kappa)\psi(\varsigma_1) + h(1-\kappa)\psi(\varsigma_2).\end{aligned}$$

□

Now we makes some examples via newly introduce definition n–polynomial s–type p–convex function.

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Example 1. *If $\psi(x) = x^p$ is p–convex function for all positive values of x and $p \in (-\infty, 0) \cup [1, \infty)$ [15], then by using Proposition 1, it is s–type p–convex function.*

Example 2. *Let $\psi : (0, \infty) \to \mathbb{R}$, $\psi(x) = x^{-p}$, $p \geq 1$, then ψ is p–convex function [15], so by using Proposition 1, it is s–type p–convex function.*

Example 3. *Let $\psi : (0, \infty) \to \mathbb{R}$, $\psi(x) = -\ln x$ and $p \geq 1$, then ψ is p–convex function [15], so by using Proposition 1, it is s–type p–convex function.*

These are the clear advantages of the proposed new definition with respect to other known functions on the topic mentioned above. Now, we will study some of its algebraic properties.

Theorem 1. *Let $\psi, \varphi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$. If ψ and φ are two s–type p–convex functions, then*

1. *$\psi + \varphi$ is s–type p–convex function.*
2. *For nonnegative real number c, $c\psi$ is s–type p–convex function.*

Proof. (1) Let ψ and φ be s–type p–convex, then

$$\begin{aligned}
&(\psi + \varphi)\left(\left[\kappa{\varsigma_1}^p + (1-\kappa){\varsigma_2}^p\right]^{\frac{1}{p}}\right) \\
&= \psi\left(\left[\kappa{\varsigma_1}^p + (1-\kappa){\varsigma_2}^p\right]^{\frac{1}{p}}\right) + \varphi\left(\left[\kappa{\varsigma_1}^p + (1-\kappa){\varsigma_2}^p\right]^{\frac{1}{p}}\right) \\
&\leq [1 - (s(1-\kappa))]\psi(\varsigma_1) + [1 - (s\kappa)]\psi(\varsigma_2) \\
&+ [1 - (s(1-\kappa))]\varphi(\varsigma_1) + [1 - (s\kappa)]\varphi(\varsigma_2)
\end{aligned}$$

$$\begin{aligned}
&= [1 - (s(1-\kappa))]\left[\psi(\varsigma_1) + \varphi(\varsigma_1)\right] + [1 - (s\kappa)]\left[\psi(\varsigma_2) + \varphi(\varsigma_2)\right] \\
&= [1 - (s(1-\kappa))](\psi + \varphi)(\varsigma_1) + [1 - (s\kappa)](\psi + \varphi)(\varsigma_2).
\end{aligned}$$

(2) Let ψ be s–type p–convex function, then

$$\begin{aligned}
&(c\psi)\left(\left[\kappa{\varsigma_1}^p + (1-\kappa){\varsigma_2}^p\right]^{\frac{1}{p}}\right) \\
&\leq c\Big[[1 - (s(1-\kappa))]\psi(\varsigma_1) + [1 - (s\kappa)]\psi(\varsigma_2)\Big] \\
&= [1 - (s(1-\kappa))]c\psi(\varsigma_1) + [1 - (s\kappa)^i]c\psi(\varsigma_2) \\
&= [1 - (s(1-\kappa))](c\psi)(\varsigma_1) + [1 - (s\kappa)](c\psi)(\varsigma_2),
\end{aligned}$$

which completes the proof. □

Theorem 2. *Let $\varphi : I \to J$ be p–convex function and $\psi : J \to \mathbb{R}$ is non-decreasing and s–type convex function. Then the function $\psi \circ \varphi : I \to \mathbb{R}$ is s–type p–convex.*

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Proof. For all $\varsigma_1, \varsigma_2 \in I$, and $\kappa \in [0,1]$, we have

$$\begin{aligned}
&(\psi \circ \varphi)\left(\left[\kappa {\varsigma_1}^p + (1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right) \\
&= \psi\left(\varphi\left(\left[\kappa {\varsigma_1}^p + (1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right)\right) \\
&\le \psi\left(\kappa\varphi(\varsigma_1) + (1-\kappa)\,\varphi(\varsigma_2)\right) \\
&\le [1-(s(1-\kappa))]\psi(\varphi(\varsigma_1)) + [1-(s\kappa)]\psi(\varphi(\varsigma_2)) \\
&= [1-(s(1-\kappa))]\,(\psi\circ\varphi)(\varsigma_1) + [1-(s\kappa)]\,(\psi\circ\varphi)(\varsigma_2),
\end{aligned}$$

which completes the proof. □

Theorem 3. *Let $\psi_i : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be an arbitrary family of s–type p–convex functions and let $\psi(\varsigma) = \sup_i \psi_i(\varsigma)$. If $O = \{\varsigma \in [\varsigma_1, \varsigma_2] : \psi(\varsigma) < +\infty\} \neq \emptyset$, then O is an interval and ψ is s–type p–convex function on O.*

Proof. For all $\varsigma_1, \varsigma_2 \in O$ and $\kappa \in [0,1]$, then we have

$$\begin{aligned}
&\psi\left(\left[\kappa {\varsigma_1}^p + (1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right) \\
&= \sup_i \psi_i\left(\left[\kappa {\varsigma_1}^p + (1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right) \\
&\le [1-(s(1-\kappa))]\sup_j \psi_j(\varsigma_1) + [1-(s\kappa)]\sup_j \psi_j(\varsigma_2) \\
&= [1-(s(1-\kappa))]\psi(\varsigma_1) + [1-(s\kappa)]\psi(\varsigma_2) < +\infty,
\end{aligned}$$

which completes the proof. □

3. Hermite–Hadamard type inequality for s–type p–convex functions

The purpose of this portion is to derive a new inequality of Hermite–Hadamard type for the s–type p–convex function ψ.

Theorem 4. *Let $\psi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be n–polynomial s–type p–convex function. If $\psi \in L_1([\varsigma_1, \varsigma_2])$, then*

$$\frac{1}{2-s}\psi\left(\left[\frac{\varsigma_1^p + \varsigma_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{\varsigma_2^p - \varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2} \frac{\psi(x)}{x^{1-p}}dx \le \Big[\psi(\varsigma_1) + \psi(\varsigma_2)\Big]\left[\frac{2-s}{2}\right]. \tag{3.1}$$

Proof. Since ψ is s–type p–convexity, we have

$$\psi\left(\left[\kappa x^p + (1-\kappa)\,y^p\right]^{\frac{1}{p}}\right) \le [1-(s(1-\kappa))]\psi(x) + [1-(s\kappa)]\psi(y), \tag{3.2}$$

which lead to

$$\psi\left(\left[\frac{x^p + y^p}{2}\right]^{\frac{1}{p}}\right) \le [1-(\frac{s}{2})]\psi(x) + [1-(\frac{s}{2})]\psi(y)$$

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Using the change of variables, we get

$$\psi\left(\left[\frac{\varsigma_1^p+\varsigma_2^p}{2}\right]^{\frac{1}{p}}\right) \le [1-(\frac{s}{2})]$$
$$\times\left\{\psi\left(\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right)+\psi\left(\left[(1-\kappa){\varsigma_1}^p+\kappa{\varsigma_2}^p\right]^{\frac{1}{p}}\right)\right\}.$$

Integrating the above inequality with respect to κ on $[0,1]$, we obtain

$$\frac{1}{2-s}\psi\left[\frac{\varsigma_1^p\varsigma_2^p}{\varsigma_1^p+\varsigma_2^p}\right]^{\frac{1}{p}} \le \frac{p}{\varsigma_2^p-\varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2}\frac{\psi(x)}{x^{1-p}}dx,$$

which completes the left side inequality. For the right side inequality, changing the variable of integration as $x=\left(\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right)$ and using the definition of the s–type p–convex function ψ, we obtain

$$\begin{aligned}
&\frac{p}{\varsigma_2^p-\varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2}\frac{\psi(x)}{x^{1-p}}dx\\
&=\int_0^1\psi\left(\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right)d\kappa\\
&\le\int_0^1\Big[[1-(s(1-\kappa))]\psi\left(\varsigma_1\right)+[1-(s\kappa)]\psi\left(\varsigma_2\right)\Big]d\kappa\\
&=\psi\left(\varsigma_1\right)\int_0^1[1-(s(1-\kappa))]d\kappa+\psi\left(\varsigma_2\right)\int_0^1[1-(s\kappa)]d\kappa\\
&=\Big[\psi\left(\varsigma_1\right)+\psi\left(\varsigma_2\right)\Big]\left[\frac{2-s}{2}\right],
\end{aligned}$$

which give the right side inequality. □

Remark 2. (*i*) *If we put* $p=1$ *in Theorem 4, then*

$$\frac{1}{2-s}\psi\left[\frac{\varsigma_1\varsigma_2}{\varsigma_1+\varsigma_2}\right] \le \frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(x)dx \le \Big[\psi\left(\varsigma_1\right)+\psi\left(\varsigma_2\right)\Big]\left[\frac{2-s}{2}\right].$$

4. Refinements of (H–H) type inequality via s–type p–convex functions

Let us recall the following Lemma that we will used in the sequel.

Lemma 2. [12] *Let* $\psi: I\to\mathbb{R}$ *be differentiable function on* I° *with* $\varsigma_1,\varsigma_2\in I$ *and* $\varsigma_1<\varsigma_2$. *If* $\psi'\in L_1[\varsigma_1,\varsigma_2]$, *then*

$$\frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2}-\frac{p}{\varsigma_2^p-\varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2}\frac{\psi(x)}{x^{1-p}}dx=\left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right) \tag{4.1}$$

$$\times\int_0^1\frac{1-2\kappa}{\left(\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]\right)^{1-\frac{1}{p}}}\psi'\left(\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right)d\kappa.$$

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Theorem 5. *Let $\psi : I \to \Re$ be differentiable function on I° with $\varsigma_1, \varsigma_2 \in I$ and $\varsigma_1 < \varsigma_2$. If $\psi' \in L_1[\varsigma_1, \varsigma_2]$ and $|\psi'|^q$ is s–type p–convex on $[\varsigma_1, \varsigma_2]$ for $q \geq 1$, then*

$$\left|\frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2} - \frac{p}{\varsigma_2^p - \varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2} \frac{\psi(x)}{x^{1-p}}dx\right| \leq \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)(B_1)^{1-\frac{1}{q}} \tag{4.2}$$

$$\times\left[B_2|\psi'(\varsigma_1)|^q + B_3|\psi'(\varsigma_2)|^q\right]^{\frac{1}{q}},$$

where

$$B_1 = \int_0^1 \frac{|1-2\kappa|}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}d\kappa, \quad B_2 = \int_0^1 \frac{|1-2\kappa|[1-(s(1-\kappa)]}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}d\kappa,$$

$$B_3 = \int_0^1 \frac{|1-2\kappa|[1-(s\kappa)]}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}d\kappa.$$

Proof. Applying Lemma 2, properties of modulus, power mean inequality and s–type p–convexity of $|\psi'|^q$, we have

$$\left|\frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2} - \frac{p}{\varsigma_2^p - \varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2} \frac{\psi(x)}{x^{1-p}}dx\right|$$
$$\leq \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)\int_0^1 \left|\frac{1-2\kappa}{\left(\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]\right)^{1-\frac{1}{p}}}\right|\left|\psi'\left(\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{\frac{1}{p}}\right)\right|d\kappa$$
$$\leq \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\kappa|}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}d\kappa\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \frac{|1-2\kappa|}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}\left|\psi'\left(\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\kappa|}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}d\kappa\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1 \frac{|1-2\kappa|}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\,\varsigma_2{}^p\right]^{1-\frac{1}{p}}}\right.$$
$$\left.\times\left\{[1-(s(1-\kappa))]|\psi\left(\varsigma_1\right)|^q + [1-(s\kappa)]|\psi\left(\varsigma_2\right)|^q\right\}d\kappa\right]^{\frac{1}{q}}$$

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$$= \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)\left(\int_0^1 \frac{|1-2\kappa|}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\varsigma_2{}^p\right]^{1-\frac{1}{p}}} d\kappa\right)^{1-\frac{1}{q}}$$

$$\times \left[\int_0^1 \frac{|1-2\kappa|[1-(s(1-\kappa))]}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\varsigma_2{}^p\right]^{1-\frac{1}{p}}} |\psi'(\varsigma_1)|^q d\kappa\right.$$

$$\left. + \int_0^1 \frac{|1-2\kappa|[1-(s\kappa)]}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\varsigma_2{}^p\right]^{1-\frac{1}{p}}} |\psi'(\varsigma_2)|^q d\kappa\right]^{\frac{1}{q}}$$

$$= \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)(B_1)^{1-\frac{1}{q}}$$

$$\times \left[B_2|\psi'(\varsigma_1)|^q + B_3|\psi'(\varsigma_2)|^q\right]^{\frac{1}{q}},$$

which completes the proof. □

Theorem 6. *Let $\varphi : I \to \mathbb{R}$ be differentiable function on I° with $\varsigma_1, \varsigma_2 \in I$ and $\varsigma_1 < \varsigma_2$. If $\varphi' \in L_1[\varsigma_1, \varsigma_2]$ and $|\varphi'|^q$ is s–type p–convex on $[\varsigma_1, \varsigma_2]$ for $q > 1$ and $\frac{1}{l} + \frac{1}{q} = 1$, then*

$$\left|\frac{\varphi(\varsigma_1)+\varphi(\varsigma_2)}{2} - \frac{p}{\varsigma_2^p - \varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2} \frac{\varphi(x)}{x^{1-p}}dx\right| \le \left(\frac{\varsigma_2^p - \varsigma_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}} \tag{4.3}$$

$$\times \left[B_4|\varphi'(\varsigma_1)|^q + B_5|\varphi'(\varsigma_2)|^q\right]^{\frac{1}{q}},$$

where

$$B_4 = \int_0^1 \frac{[1-(s(1-\kappa))]}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\varsigma_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\kappa$$

and

$$B_5 = \int_0^1 \frac{[1-(s\kappa)]}{\left[\kappa\varsigma_1{}^p + (1-\kappa)\varsigma_2{}^p\right]^{q\left(1-\frac{1}{p}\right)}} d\kappa.$$

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Proof. Applying Lemma 2,properties of modulus, Hölder's inequality and n–polynomial s–type p–convexity of $|\varphi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2}-\frac{p}{\varsigma_2^p-\varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2}\frac{\psi(x)}{x^{1-p}}dx\right| \\
&\leq \left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\left(\int_0^1|1-2\kappa|^l d\kappa\right)^{\frac{1}{l}} \\
&\times\left(\int_0^1\frac{1}{\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{q(1-\frac{1}{p})}}\left|\psi'\left(\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{\frac{1}{p}}\right)\right|^q d\kappa\right)^{\frac{1}{q}} \\
&\leq \left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}} \\
&\times\left(\int_0^1\frac{[1-(s(1-\kappa))]|\psi\left(\varsigma_1\right)|^q+[1-(s\kappa)]|\psi\left(\varsigma_2\right)|^q}{\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{q\left(1-\frac{1}{p}\right)}}d\kappa\right)^{\frac{1}{q}} \\
&= \left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\left(\frac{1}{1+l}\right)^{\frac{1}{l}} \\
&\times\left[B_4|\psi'(\varsigma_1)|^q B_5|\psi'(\varsigma_2)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 7. *Let $\varphi : I \to \mathbb{R}$ be differentiable function on I° with $\varsigma_1, \varsigma_2 \in I$ and $\varsigma_1 < \varsigma_2$. If $\varphi' \in L_1[\varsigma_1, \varsigma_2]$ and $|\varphi'|$ is s–type p–convex on $[\varsigma_1, \varsigma_2]$, then*

$$\left|\frac{\varphi(\varsigma_1)+\varphi(\varsigma_2)}{2}-\frac{p}{\varsigma_2^p-\varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2}\frac{\varphi(x)}{x^{1-p}}dx\right|\leq\left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right) \tag{4.4}$$

$$\times\left(B_6|\varphi'(\varsigma_1)|^q+B_7|\varphi'(\varsigma_2)|^q\right),$$

where

$$\begin{aligned}
B_6 &= \int_0^1\frac{|1-2\kappa|[1-(s(1-\kappa))]}{\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{1-\frac{1}{p}}}d\kappa, \\
B_7 &= \int_0^1\frac{|1-2\kappa|[1-(s\kappa)]}{\left[\kappa{\varsigma_1}^p+(1-\kappa)\,{\varsigma_2}^p\right]^{1-\frac{1}{p}}}d\kappa.
\end{aligned}$$

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Proof. Applying Lemma 2, properties of modulus and s–type p–convexity of $|\varphi'|^q$, we have

$$\left|\frac{\varphi(\varsigma_1)+\varphi(\varsigma_2)}{2}-\frac{p}{\varsigma_2^p-\varsigma_1^p}\int_{\varsigma_1}^{\varsigma_2}\frac{\varphi(x)}{x^{1-p}}dx\right|$$

$$\leq\left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\int_0^1\left|\frac{1-2\kappa}{\left(\left[\kappa{\varsigma_1}^p+(1-\kappa){\varsigma_2}^p\right]\right)^{1-\frac{1}{p}}}\right|\left|\varphi'\left(\left[\kappa{\varsigma_1}^p+(1-\kappa){\varsigma_2}^p\right]^{\frac{1}{p}}\right)\right|d\kappa$$

$$\leq\left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\int_0^1\left|\frac{1-2\kappa}{\left(\left[\kappa{\varsigma_1}^p+(1-\kappa){\varsigma_2}^p\right]\right)^{1-\frac{1}{p}}}\right|$$

$$\times\left[[1-(s(1-\kappa))]|\varphi'(\varsigma_1)|+[1-(s\kappa)]|\varphi'(\varsigma_2)|\right]d\kappa$$

$$=\left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\left[|\varphi'(\varsigma_1)|\int_0^1\frac{|1-2\kappa|[1-(s(1-\kappa))]}{\left[\kappa{\varsigma_1}^p+(1-\kappa){\varsigma_2}^p\right]^{1-\frac{1}{p}}}d\kappa\right.$$

$$\left.+|\varphi'(\varsigma_2)|\int_0^1\frac{|1-2\kappa|[1-(s\kappa)]}{\left[\kappa{\varsigma_1}^p+(1-\kappa){\varsigma_2}^p\right]^{1-\frac{1}{p}}}d\kappa\right]$$

$$=\left(\frac{\varsigma_2^p-\varsigma_1^p}{2p}\right)\left(B_6|\varphi'(\varsigma_1)|+B_7|\varphi'(\varsigma_2)|\right),$$

which completes the proof. □

5. Conclusion

We have introduced and investigated some algebraic properties of a new class of functions namely s–type p–convex. We showed that this class of functions had some nice properties, which other convex functions had as well. We proved that our new class of s–type p–convex function is very larger with respect to the known class of functions, like convex and harmonically convex. New version of Hermite–Hadamard type inequality and an integral identity for the differentiable functions are obtained. In recent years, many researchers put the effort into the theory of inequalities to bring a new dimension to mathematical analysis and applied mathematics with different features in the literature. Due to widespread views and applications, the theory of inequalities has become an attractive, interesting and absorbing field for the researchers. It is high time to find the applications of these inequalities along with efficient numerical methods. The interested reader can found other new results using other suitable functions ψ and also new bounds for special means and error estimates for the trapezoidal and midpoint formula. To the best of our knowledge these results are new in the literature. Since convex functions has large applications in many mathematical areas, we hope that our new results can be applied in convex analysis, special functions, quantum analysis, quantum mechanics, post quantum analysis, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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